SIAM J. APPL. MATH.
Vol. 28, No. 1, January 1975

SOLUTIONS OF THE NAVIER–STOKES EQUATION AT LARGE REYNOLDS NUMBER*

PACO A. LAGERSTROM†

Abstract. The problem of two-dimensional incompressible laminar flow past a bluff body at large Reynolds number (R) is discussed. The governing equations are the Navier–Stokes equations. For $R = \infty$, the Euler equations are obtained. A solution for R large should be obtained by a perturbation of an Euler solution. However, for given boundary conditions, the Euler solution is not unique. The solution to be perturbed is the relevant Euler solution, namely the one which is the Euler limit of the Navier–Stokes solution with the same boundary conditions. For certain semi-infinite or streamlined bodies, the relevant Euler solution represents potential flow. For flow inside a closed domain a theorem of Prandtl states the relevant Euler solution has constant vorticity in each vortex. In many cases it can be determined by simultaneously considering the boundary layer equations. For flow past a bluff body, the relevant Euler solution is not known, although the free streamline flow for which the free streamline detaches smoothly from the body is a likely candidate. Even if this is correct, many unsolved problems remain. Various scalings have to be used for various regions of the flow. Possibilities of scaling for the various regions are discussed here. Special attention is paid to the region near the point of separation. A famous paper by Goldstein asserts that for an adverse smooth pressure gradient, the solution of the boundary layer equations can, in general, not be continued beyond the point of separation. Subsequent attempts by many authors to overcome the difficulty of continuation have failed. A very promising theory, going beyond conventional boundary layer theory, has recently been put forward independently by Sychev and Messiter. They assume that separation takes place in a sublayer whose thickness and length tend to zero as R tends to infinity. The pressure gradient in the sublayer is self-induced and is positive upstream of the point of separation and zero downstream. Their theory does not contradict experiments and numerical calculations, which may be reliable up to, say, $R = 100$, but it also shows that in this context, 100 may not be regarded as a large Reynolds number. The sublayer has the same scaling in orders of R as the sublayer at the trailing edge of a plate, found earlier by Stewartson and Messiter in studying the matching of the boundary layer solution on the plate with the Goldstein wake solution downstream of the trailing edge.

In this paper we shall discuss solutions of the Navier–Stokes equations for large Reynolds number (low viscosity). We shall restrict ourselves to the case of two-dimensional, incompressible, stationary flow. The last restriction implies that the solutions do not describe real flow, but that we are dealing with unstable solutions and that the real flow, for the same boundary conditions, is unsteady. Still, the author believes that investigations of stationary solutions, besides presenting challenging mathematical problems, eventually will shed light on the behavior of real flow.

The Navier–Stokes equations are, for the restricted case we consider here,

$$\rho\left(u_d\frac{\partial u_d}{\partial x_d} + v_d\frac{\partial u_d}{\partial y_d}\right) + \frac{\partial p_d}{\partial x_d} = \mu\left(\frac{\partial^2 u_d}{\partial x_d^2} + \frac{\partial^2 u_d}{\partial y_d^2}\right), \tag{1a}$$

$$\rho\left(u_d\frac{\partial v_d}{\partial x_d} + v_d\frac{\partial v_d}{\partial y_d}\right) + \frac{\partial p_d}{\partial y_d} = \mu\left(\frac{\partial^2 v_d}{\partial x_d^2} + \frac{\partial^2 v_d}{\partial y_d^2}\right) \tag{1b}$$

* Received by the editors June 17, 1974. Presented by invitation at an International Symposium on Modern Developments in Fluid Dynamics in Honor of the 70th Birthday of Sydney Goldstein held at Haifa, Israel, December 16–23, 1973. The research was supported in part by the National Science Foundation under Grant GP 28129.

† Department of Applied Mathematics, California Institute of Technology, Pasadena, California 91109.

$$\frac{\partial u_d}{\partial x_d} + \frac{\partial v_d}{\partial y_d} = 0. \tag{1c}$$

Here x_d, y_d are Cartesian coordinates of dimension length; u_d and v_d are the corresponding velocity components; ρ is density and μ is viscosity, both considered constant. If the boundary conditions provide a characteristic velocity U, a characteristic geometric length L (for a conical body we only have the viscous length $\lambda = \mu/(\rho U)$) and a characteristic pressure P, we may introduce the nondimensional variables

$$x = \frac{x_d}{L}, \quad y = \frac{y_d}{L}, \quad u = \frac{u_d}{U}, \quad v = \frac{v_d}{U}, \quad p = \frac{p_d - P}{\rho U^2} \tag{2a}$$

and the nondimensional parameter

$$R = \text{Reynolds number} = \rho UL/\mu = L/\lambda. \tag{2b}$$

The nondimensional form of (1) is then

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial p}{\partial x} = \frac{1}{R}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{3a}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{\partial p}{\partial y} = \frac{1}{R}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right), \tag{3b}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{3c}$$

The corresponding Euler equations are obtained by putting $\mu = 0$ in (1) or $R = \infty$ in (3).

Finding a solution for large Reynolds numbers implies that we want to find asymptotic formulas valid as R tends to infinity, in other words, that we deal with a perturbation problem. There are several general aspects of the special perturbation problem we are studying.

1. We are dealing with one physical theory, that of nonviscous flow governed by the Euler equations, as an approximation to another physical theory, that of viscous flow governed by the Navier–Stokes equations.

2. Solutions of the Euler equations admit discontinuities in velocity from one streamline to an adjacent streamline, although the pressure must be continuous. Such discontinuities are consistent with the theory of characteristics and with the integral conservation laws whose differential form are the Euler equations. Since such discontinuities may not occur in Navier–Stokes solutions, we expect the perturbation problem to be singular. As boundary conditions for the viscous flow, we shall require that at the surface of a solid the normal velocity component q_n be zero (no suction or blowing) and that the tangential velocity component q_t be given. For flow outside a solid, we shall restrict ourselves to the condition $q_t = 0$ at the surface of the solid. For the corresponding Euler flow, we may as well keep the same conditions, but we expect the tangential velocity at the solid to jump discontinuously from the given q_t to some a priori unknown value.

3. The solutions of the Euler equations for given boundary conditions are not unique.

The first point made above is that we are dealing with the relations between a less accurate theory and more accurate theory. In some significant limit, solutions of the second theory must tend to solutions of the first theory. Conversely, we expect to obtain asymptotic solutions of the second theory by perturbating some relevant solution of the first theory. There are abundant examples of such pairs of theories: theory of incompressible flow (viscous or not) and theory of compressible flow; Newton's gravitational theory and Einstein's gravitational theory, etc. In all such cases, we believe that the equations for the less accurate theory are obtained by putting a certain physical parameter (or parameters) equal to zero in the equations for the more accurate theory. Correspondingly, we assume that for the same conditions (boundary, initial, etc.) a solution of the more accurate equations becomes a solution of the less accurate equations by letting the relevant parameter tend to zero. In our case, the relevant parameter is μ. As stated above, the Euler equations are obtained from the Navier–Stokes equations by putting $\mu = 0$. We then also expect that a solution of the Navier–Stokes equations tends to a solution of the Euler equations by putting μ equal to zero. In the nondimensional form of the equations, this relevant limit is letting R^{-1} tend to zero, using a scaling of variables such as given by (2a) which does not involve μ. We shall call this limit the *Euler limit.*[1]

The second point made above implies that the Euler limit of a Navier–Stokes solution may not be uniformly valid. To obtain a uniformly valid perturbation, we may have to use several different scalings. In his paper [1] of 1905, Prandtl introduced boundary layer theory. For example, near the surface of a flat plate parallel to the flow at infinity, Prandtl introduced the rescaled variables $\tilde{y} = R^{1/2}y$ and $\tilde{v} = R^{1/2}v$ (assuming $y = 0$ at the plate). Introducing these variables into (3) and then letting R^{-1} formally tend to zero, one obtains the Prandtl boundary layer equations. It has since been discovered that, in realistic cases, many different scalings will be needed for various regions in addition to the Prandtl scaling and the scaling given by (2a). Still, if the Euler limit is applied to various rescaled expressions, valid in different regions, an Euler solution for the entire flow must be obtained, even though it may exhibit discontinuities consistent with the Euler equations.

The third point made above presents a puzzle which does not seem to occur in other pairs of less accurate versus more accurate theories. For instance, Newton's gravitational equations give, in principle, unique solutions for the motion of the planets in the solar system (which incidentally check observations except very precise observations of the orbit of Mercury). In our case, the Euler solution is not unique. We shall call a solution of the Euler equations, which is the Euler limit of the Navier–Stokes solution for the same conditions, a *relevant* Euler solution. This solution may not be known a priori, but may have to be determined by considering a consistent perturbation solution, using different scalings. Still, under the Euler limit, the Navier–Stokes solution must become an Euler solution, and it is, in principle, this relevant Euler solution which is being perturbed. The nonuniqueness of the Euler solution for given conditions is an old

[1] We are making the assumption that for the boundary conditions dealt with here the Navier–Stokes solution exists and is unique for all R.

problem. Assuming that the vorticity is everywhere zero—that is, potential flow—leads to a solution for flow past a solid which predicts zero drag (Lagrange's paradox). A solution, or rather a one-parameter class of solution which predicts nonzero drag, is given by the Helmholtz–Kirchhoff free streamline solution which is potential except at the free streamline, where the vorticity is infinite. Comparison with experiments is, of course, obscured by the instability of the flow at large values of R. There are other possible Euler solutions for the flow past a solid. However, these have not been considered until recently. This problem of relevant Euler solutions will be discussed in detail below.

The type of perturbation theory described above should be contrasted with another type of perturbation problems. Within one physical theory, we perturb a solution with simple conditions to obtain a solution for slightly less simple conditions. Examples are: solutions of the restricted three-body problem within the framework of Newtonian gravitational theory, solutions for flow past slender bodies within the framework of Euler theory, assuming the special case of Euler flow which has zero vorticity (potential flow). Another example is flow at high viscosity (low Reynolds number). By an analysis and extension of Prandtl's boundary layer theory, Kaplun found ([2]; see also [3]) that these ideas may be used to explain the Stokes and Oseen equations. The flow to be perturbed is uniform flow, which represents first term of the outer expansion. If a very small body is inserted into this flow, this outer solution is not uniformly valid to order unity near the body. By changing the scaling, one obtains the Stokes equations which are valid near the body for $R \ll 1$. A solution of these equations which matches with the first term of the outer expansion gives the first term of the inner expansion. This term, in its turn, determines a second term of the outer expansion obeying the Oseen equations, which, in turn, determines a second term of the inner solution, etc. From a historical point of view, it is interesting to note that Prandtl's ideas (often only partially valid for large R) when properly analyzed led to a solution for small R. From a logical point of view, we note the *hierarchy* of the various terms. First, one determines the first term of the outer expansion, then the first term of the inner expansion, then the second term of the outer expansion, etc. So-called switch-back terms do not upset this hierarchy if the partial inner and outer expansions are determined by their domains of validity rather than as limit process expansions. This point of view is stated somewhat elliptically in [2] and discussed in detail in [4].

We emphasize again the fundamental difference in principle between the cases of small viscosity and large viscosity. There is a physical theory, expressed by the Euler equations, which assumes that viscosity is zero. An asymptotic expansion for small μ must tend to an Euler solution in the limit μ tending to zero. There is no physical theory which states that viscosity is infinite. The governing equations for large values of μ are still the Navier–Stokes equations. Dimensional analysis shows that the condition of μ tending to infinity may be replaced by L (characteristic body length) tending to zero, everything else being fixed. The first term of the outer expansion, uniform flow, obeys the Navier–Stokes equations. The first term in the inner expansion obeys the Stokes equations which are comparable to Prandtl's boundary layer equations. The Stokes equations do not represent a physical theory in the sense that the Euler equations do, and they do not contain the Oseen equations which govern the second term of the outer expansion. Neither do the Oseen equations represent a physical theory.

We now return to flow and large Reynolds number. As a first example, we take the case of flow inside a closed domain. In [1] Prandtl derived the result that in the limit of vanishing viscosity, the vorticity is constant in each vortex (defined as a maximum set of nested closed streamlines). It is interesting to review the essence of Prandtl's derivation of his theorem. It depends on two formulas. Defining

$$\text{vorticity} = \omega = \partial v/\partial x - \partial u/\partial y \tag{4a}$$

and

$$\omega_0 = \lim_{\mu \to 0} \omega, \tag{4b}$$

we find that for Euler flow,

$$u\frac{\partial \omega_0}{\partial x} + v\frac{\partial \omega_0}{\partial y} = 0, \tag{5a}$$

and that for Navier–Stokes flow, for an arbitrary R,

$$\frac{1}{R}\oint \frac{\partial \omega}{\partial n}\, ds = 0 \tag{5b}$$

along a closed streamline. In the last equation, we may multiply by R and obtain in the limit of R tending to infinity,

$$\oint \frac{\partial \omega_0}{\partial n}\, ds = 0, \tag{6}$$

along a closed streamline. We note that (5a) is valid for any Euler solution, whereas (6) cannot be derived from the Euler equations; it is, in fact, false for an arbitrary Euler solution. It is valid only for a *relevant* Euler solution, that is, for an Euler solution which is the Euler limit of a Navier–Stokes solution.

To fix the ideas, let us consider flow inside a circle of unit radius ($x^2 + y^2 = 1$ at the boundary), and let us prescribe $q_n = 0$ and q_t to be a given function of the angle θ at the boundary. There is an infinity of Euler solutions. (As remarked earlier, the value of q_t does not influence the Euler solutions, except the relevant one as explained below, since we expect a discontinuity of the tangential velocity at the boundary.) For example, we may assume the streamlines to be concentric circles, prescribe any velocity distribution which is constant on each circle, and adjust the pressure gradient to balance the centrifugal forces.

Batchelor [5] and Feynman and Lagerstrom [6] conceived the idea that the relevant Euler solution must be such that the corresponding boundary layer must be periodic. From this idea, together with the Prandtl theorem, Wood [7] and Feynman–Lagerstrom [6] derived the formula

$$\frac{1}{2\pi}\int_0^{2\pi} q_t^2\, d\theta = q_E^2, \tag{7}$$

where q_E = tangential velocity of the Euler flow as the radius of the streamline approaches unity. Since (according to Prandtl) the vorticity is constant in the limit, that is, the fluid rotates like a solid, we have

$$|\omega_0| = 2|q_E|. \tag{8}$$

This determines the *relevant* Euler solution provided q_t does not differ too much from a constant value. In this case, the sign of q_t determines the sign of q_E and of ω_0, and we have found the relevant Euler solution. However, there may be other cases for which the relevant Euler solution consists of two symmetrical vortices with vorticity of opposite sign or a combination of various vortices of unknown boundaries.

The reasoning above shows that under certain conditions on q_t, not numerically specified, the relevant Euler solution may be found. For more general values of q_t, the relevant Euler solution may consist of several vortices, often with a priori unknown boundaries.

This brings up two deficiencies in theory which have not been given due mathematical attention: (i) the possible Euler solutions of a given problem have not been investigated; (ii) the relevant Euler solution of a problem has, with few exceptions, not been investigated. For instance, the possible Euler solutions with several vortices inside a circle, each with a different constant vorticity, have, to the author's knowledge, not been determined.

Still, in our example, if q_t does not differ too much from a constant value, the relevant Euler solution can be determined. In a sense there is a loss of hierarchy. The relevant Euler solution is determined only by considering boundary layer theory. Still, we must regard the relevant Euler solution as the principal solution. It is this solution which is being perturbed.

We shall now discuss external flow past solids. We shall assume uniform flow at infinity, which gives the characteristic velocity U, and we take the constant pressure at infinity as our characteristic pressure P. At the surface of the solid, we assume the no-slip condition, i.e., both tangential and normal velocity components vanish.

Singular perturbation techniques must be used to find asymptotic expansion. There have been isolated instances of use of singular perturbation techniques at least as far back as Laplace. However, Prandtl's paper on boundary layer theory [1] really put the technique, or rather the rudiments of the technique, on the map. After Prandtl's original paper, attention was paid mainly to solutions of special cases of the boundary layer equations. A critical study of their meaning was done only rarely, outstanding exceptions being papers by Goldstein (to be discussed later). It was only in the 1950's that the relation of boundary layer theory to asymptotic expansion was studied systematically, leading to what is now known as the method of matched asymptotic expansions.

Prandtl's boundary layer equations are valid in a thin layer near the body. In particular, for the case of a semi-infinite flat plate, he showed how these equations reduce to what later became known as the Blasius equation. The semi-infinite flat plate, like the semi-infinite wedge studied by Falkner–Skan, has no Reynolds number. In the mathematical sense, viscosity is here an artificial parameter (see [8] and [9]). As a consequence, the expansion is really one for large values of the nondimensional coordinate $x_d U\rho/\mu$ rather than in the nonexisting Reynolds number. However, the method of matched asymptotic expansions may still be used (see Goldstein [10]). The main difference is the occurrence of constants which cannot be determined by this method. This problem is discussed in detail in Chang [9]. A similar situation occurs in the determination of the flow at large distances from a finite body at a fixed R (see Goldstein [11], and for a detailed discussion, Chang [9]).

The outstanding problem is that of separation, a concept to be discussed and made precise later. For certain semi-infinite bodies, one may expect that there is no point of separation. The class cannot be precisely defined, but the parabola seems to be one example. In this case, it is reasonable to assume that the relevant Euler solution is that of potential flow and, in principle, the problem can be solved. In such a problem there is a definite hierarchy: the first outer solution, potential flow, is determined without considering the boundary layer solution. From this solution one determines the boundary layer solution, which determines the second outer solution. This second term of the outer expansion is normally called flow due to displacement thickness, although a better term is potential flow due to an apparent source distribution, since the latter concept can easily be generalized to three dimensions and to compressible flow. Then one may determine a correction to the boundary layer flow (second inner solution), etc. Calculations have been carried out for the parabola by Van Dyke [12]. It is easily shown that for *incompressible* flow, all terms of the outer expansion represent potential flow.

Another class of problems for which separation is not expected and for which the relevant Euler solution is potential flow past thin bodies with rounded leading edge and cusped trailing edge at moderate angles of attack (airfoils). The potential solution is not unique, since a potential vortex of arbitrary strength may be placed in the nonphysical region inside the body. The strength of this vortex is usually determined by the Kutta–Joukowski condition. It is reasonable to assume that this condition determines the relevant Euler solution, although no proof exists of this assumption.

A limiting case of an airfoil is a finite flat plate (of length L) placed parallel to the free stream, say along the x-axis. The limiting Euler solution obviously represents uniform flow, and near the plate, except at leading and trailing edges, boundary layer theory is valid. Goldstein [13] determined the wake flow in a thin wake region downstream of the trailing edge (which we take as $x = 0$). The first term in the Goldstein solution gives the stream function ψ as

$$\tilde{\psi} = R^{1/2}\psi(x, R^{1/2}y) \sim x^{2/3} f_0(\eta), \qquad x \geqq 0, \tag{9a}$$

$$\eta = R^{1/2}y/x^{1/3}. \tag{9b}$$

Goldstein found a similarity solution for f_0 and also determined higher order terms in x (which is considered small). In a certain limit, the Goldstein wake solution matches the boundary layer solution at $x = 0$. The wake solution represents a thin sublayer for x small and positive. It influences the continuation of the main boundary layer for $x > 0$, the form of which is also determined.

It was discovered independently by Stewartson [14] and Messiter [15] that the matching is not complete. In particular, the pressure due to displacement thickness would be discontinuous at $x = 0$. The authors quoted therefore proposed a layer near $x = 0$ which is thin relative to the main boundary layer and whose extent in the x-direction is infinitesimal, i.e., tends to zero as R tends to infinity.

The author believes that the Stewartson–Messiter discovery is of great importance beyond its original context. In abstract terms, they solved the following problem. We try to scale the independent and dependent variables such that:

(i) after the rescaled variables are introduced, one obtains a sublayer for which the full boundary layer equations are valid;

(ii) the sublayer must match with the normal boundary layer—in particular, for large values of the rescaled y, the sublayer must have a linear profile;

(iii) the term $\partial p/\partial x$ in the equations must be due to displacement thickness computed for the normal boundary layer, under the assumption that it is simply lifted up by the sublayer and that $\partial p/\partial y$ remains zero.

These conditions uniquely determine the following orders of magnitude:

$$x \sim R^{-3/8}, \quad y \sim R^{-5/8}, \quad u \sim R^{-1/8},$$
$$v \sim R^{-3/8}, \quad p \sim R^{-1/4}, \quad \frac{\partial p}{\partial x} \sim R^{1/8}. \tag{10}$$

We note that we have a genuine sublayer since $R^{-5/8} \ll R^{-1/2}$ and that the pressure can be computed from linearized potential theory since the slope of the apparent body is $\sim y/x \sim R^{-1/4}$. In the limit of infinite Reynolds number the pressure gradient becomes infinite, but in a region of zero length, so that the Euler limit is still uniform flow. An important feature is that the pressure is self-induced, that is, the pressure due to displacement thickness is determined simultaneously with the revised boundary layer solution. Unlike the solution for flow past a parabola, this solution exhibits a definite loss of hierarchy.

It should be noted that at $x = 0$, the boundary condition at $y = 0$ changes from $u = 0$ to the symmetry condition $\partial u/\partial y = 0$. One would therefore expect that the term $\partial^2 u/\partial x^2$ in the Navier–Stokes equations would be important and furnish the upstream influence of the wake which results in the Stewartson–Messiter sublayer. However, it has been shown that the term $\partial^2 u/\partial x^2$ enters only when orders $R^{-3/4}$ are considered. To the order considered by Stewartson and Messiter, the mechanism for upstream influence is the pressure due to displacement thickness.

Finally, we come to the problem of flow past a bluff body and encounter the phenomenon of separation. For convenience, we shall only consider symmetric bodies, in particular, circular cylinders with center at the origin. The x-axis upstream of the body is a streamline, and we assign the value zero to the stream function ψ along this line. At the forward stagnation point, this line bifurcates: ψ is zero along the frontal surface of the body for $y \geqq 0$ and for $y \leqq 0$. We need only consider $y \geqq 0$. This bifurcation occurs also in potential theory and is well understood. When R is not too small, a second bifurcation occurs further downstream. ψ is zero along a streamline leaving the body and also zero along the body in a region of backflow. We call the first streamline the *detached streamline* and the point where it detaches the *point of detachment*. As R tends to infinity, we obtain a *limiting* point of detachment. A different concept is the point of vanishing skin friction. This is an asymptotic concept. Assume that we know the relevant Euler solution. According to this solution, all streamlines coming from upstream infinity have zero vorticity, and Prandtl's boundary layer theory is expected to be valid near the body in the potential region. The pressure along the body is then known in this region. We may then in principle solve the Prandtl equations. This solution may give a point at the surface of the body at which the skin friction vanishes, that is, $\partial q_t/\partial n = 0$, where n is the normal distance from the wall and q_t is the tangential velocity in the sense of boundary layer theory. The point of detachment (limiting or not) and the point of vanishing skin friction are loosely referred to as the *point of separation*. However, it must be emphasized that the limiting point of detachment and the point of vanishing skin friction are conceptually different even though they might actually coincide.

In [1] Prandtl associated separation with an adverse pressure gradient, i.e., a pressure increase in the downstream direction. The pressure is constant in the boundary layer in the direction normal to the body. The inner part of the boundary layer has not sufficient momentum to offset the adverse pressure gradient; it will slow down, and eventually the $\partial q_t/\partial n$ will be zero at some point, beyond which, if boundary layer theory still holds, q_t would become negative, i.e., backflow will occur. Prandtl's reasoning contained the germ of a very important idea. In developing this idea, Goldstein [16] studied the velocity profile at the point of vanishing skin friction and the upstream singularities at this point. He concluded that, in general, the boundary layer equations have no solution downstream of a point of vanishing skin friction. He assumed that near this point, the pressure gradient is given, to lowest order, by a positive constant. Goldstein's research was continued by Stewartson and others. An ingenious discussion of the main ideas as well as a complete discussion of the singularity was given by Kaplun [3]. If Goldstein's analysis is correct, the limiting point of detachment cannot occur downstream of the point of vanishing skin friction since the boundary layer thickness tends to zero in the Euler limit. The possibility exists, however, that the limiting point of detachment occurs before the skin fraction reaches the value zero. It can be shown (see [17]) that for a body with blunt leading and trailing edges, the skin friction must vanish somewhere along the body if the boundary layer flow is computed using the potential solution as outer flow. This, together with Goldstein's result, would rule out the potential solution as the relevant Euler solution in such a case. Conversely, potential flow past a parabola has an everywhere favorable pressure gradient, which is the reason that the Van Dyke solution for the parabola mentioned above is plausible.

There still remains some controversy over Goldstein's work. Stewartson [18] found that the Goldstein singularity disappears if the wall temperature differs from the temperature at infinity. On the other hand, Buckmaster [19], guided by ideas by Kaplun, found that if the wall is cold (relative to infinity), singularities occur at the point of vanishing skin friction, but found no singularity for a hot wall. He used a more general form of the expansion near the point of vanishing skin friction than Goldstein or Stewartson. In any asymptotic analysis, one assumes some form of expansion, and the assumption of too restricted a form may lead to an erroneous result, and more analytical work is needed for this problem. Independent of its original context, Goldstein's result initiated some important research in the mathematical theory of nonlinear parabolic equations.

Before continuing the theoretical discussion, we shall consider some experimental and numerical results and specialize to flow past a circular cylinder. Results exist and agree and may be considered reliable up to, say, $R = 100$. Here the characteristic length L is the diameter of the cylinder. Experimentally, the flow may be made stationary in the relevant region with the aid of a splitter plate. Experiments and calculations show the following qualitative picture: on the frontal part of the solid there is a boundary layer followed by a point of detachment. The detaching streamline, the body surface downstream of detachment and the x-axis enclose a finite wake bubble consisting of one vortex. (Because of symmetry we need only consider points with $y \geqq 0$.) The actual body plus the wake bubble form an elongated body, outside of which there are no closed streamlines. Quantitatively, the results are mainly inconclusive. The pressure at the forward stagnation point approaches a value consistent with asymptotic theory, and the length of the wake bubble seems to grow linearly with R.[2] The following important quantities move in a definite direction but do not settle down to a definite value. The point of detachment moves upstream as R increases, the pressure at the rear stagnation point is negative and increases, the (nondimensional) drag decreases. For a fixed R, the pressure is negative and increases in the flow direction in a region upstream of the point of detachment, but its gradient becomes smaller downstream. This almost flat region of pressure distribution becomes more pronounced for higher values of R. There does not seem to be any observable trend in the width of the wake bubble. The velocity seems to decrease in the interior of the wake bubble, and vorticity does not approach any constant value, except possibly zero.

For the analytical work, various regions have to be studied with various scalings and be matched to form a consistent picture. The boundary layer region is well understood, except that the pressure gradient is unknown a priori. Quite recently some progress has been made in the local study of the point of detachment. Independently, Sychev [21] and Messiter [22] have proposed a theory which in the author's view seems very promising. They observed that at the point of detachment, there is a change in boundary conditions at the detaching streamline which resembles the change of boundary conditions at the trailing edge of a flat plate. At the point of detachment, they therefore introduce a sublayer scaled as in (10) and show how a boundary layer solution upstream of this point can be joined to a shear layer around the detaching streamline. The assumed nature of the downstream singularity at this point is similar to the Goldstein singularity at the point of vanishing skin friction, although it uses different power laws. The pressure gradient is adverse upstream and zero on the detached streamline and in the thin region of backflow. This assumption differs from Goldstein's assumption of a pressure gradient which is approximately constant around the point of vanishing skin friction. Furthermore, as R increases, the point of detachment moves upstream as $R^{-1/16}$. Thus locally, an adverse pressure gradient precedes

[2] For this reason the assumption of a finite wake bubble as R tends to infinity made by Batchelor in Ref. 20, does not seem plausible.

detachment. However, in the strict Euler limit, the interaction region shrinks to zero and the flow detaches without an adverse pressure gradient.

The Sychev–Messiter solution is at least consistent with the experimental and numerical results mentioned above. Since very small negative powers of R enter, it makes it plausible that $R = 100$ is not a large Reynolds number (although in the expansion for small R, it is indeed a very large value, significantly outside the region of validity of this expansion.) For instance, if we assume $R^{-1/16}$ to be $\frac{1}{2}$, then R has to be approximately 66,000.

The Sychev–Messiter theory is also consistent with, but does not require, the assumption that the Euler limit be the Brillouin–Villat flow, i.e., the Kirchhoff free streamline solution for which the flow detaches with continuous tangent. For this flow, the streamline detaches at an angle of approximately 55°, counted from the forward stagnation point. It has been argued that such a solution cannot be the limit, since if it is perturbed for small viscosity, the resulting shear layers around the free streamlines never meet, and we do not obtain a closed wake bubble. To the author, this argument takes a very narrow view of possible perturbation scheme and is not convincing.

To obtain a global theory for large values of R, much further research is needed. Different scalings must be used for different regions, and the different pieces of the asymptotic must be matched to obtain a consistent uniformly valid asymptotic solution. No convincing scheme has as yet been presented. We shall, however, make a few general remarks.

As an example of a special region, let us consider the wake bubble. Special scalings may be needed near the solid and the point where the bubble closes (that is, where the detached streamlines reach the x-axis). In the region between the wake bubble and the outside flow, the assumption of a shear layer seems reasonable. We shall exclude these regions and consider the main part of the wake bubble. Since its width-to-length ratio presumably tends to zero as R tends to infinity, we find from the continuity equation that $v \ll u$. There are two possible cases: (i) $u = o(1)$; (ii) $u = O(1)$, that is, u remains nonzero as R tends to infinity. In the first case, applying the Euler limit, we find that velocity is zero in the wake for the relevant Euler solution. This means that the relevant Euler solution is a free streamline solution. Possibly, there could also be finite buffer vortices, each with constant vorticity, downstream of the point of detachment. Excluding this possibility, we find that only one free streamline solution is reasonable, namely the Brillouin–Villat solution for which the flow detaches tangentially. As mentioned above, the angle of the limiting point of detachment is then approximately 55° counted from the forward stagnation point. If the flow detaches at a smaller angle, the detaching streamline would enter the body. At a larger angle, the point of detachment is preceded by an adverse pressure gradient which becomes infinite at the point of detachment. This case seems excluded by the Prandtl–Goldstein analysis. Note that for Brillouin–Villat flow, the pressure gradient is favorable upstream of the point of detachment and that there is no point of vanishing skin friction in the asymptotic sense defined earlier. As stated above, this is perfectly consistent, although not required by the Sychev–Messiter analysis of flow near the point of detachment. This analysis predicts an adverse pressure gradient, which seems necessary for detachment, in a region which may seem large for $R = 100$

but which tends slowly—very slowly—to zero as R approaches infinity.

In the second case mentioned above, u remains nonzero as R tends to infinity. If we assume that the length of the wake bubble tends to infinity with R, the relevant Euler solution must exhibit an infinite wake with nonzero velocity and vorticity. The streamlines in the wake must come from downstream infinity, turn around and return to downstream infinity. The vorticity on each streamline must be constant, but may vary from streamline to streamline since Prandtl's theorem about constant vorticity has not been proven for regions whose length increase linearly with R. To the author's knowledge, such Euler solutions have not been considered. However, any theory proposing that u remains of order unity in the wake bubble must be complemented by exhibiting Euler solutions of the type just mentioned.

Some further general remarks about the possible scalings for the equations for the main part of the wake bubble can be made. Assume that, if we use the body diameter as unit length, its length[3] is of order R, its width of order R^a, $a < 1$, and that u is of order R^{-b}. In (3a) the $u\,\partial u/\partial x$ and $v\,\partial u/\partial y$ will always balance because of the continuity equation (3c). The term $\partial^2 u/\partial x^2$ will be negligible compared to $\partial^2 u/\partial y^2$. Hence the main problem in determining the proper asymptotic equations is the balance between $u\,\partial u/\partial x$ and $(1/R)(\partial^2 u/\partial y^2)$. If $2a < b$, the second term dominates, which leads to unacceptable equations. If $2a = b$, we obtain the boundary layer equations, which retain the term $(1/R)(\partial^2 u/\partial y^2)$ in (3a). (This does not mean that the flow remains viscous in the limit of R tending to infinity: in the Euler limit all terms involving viscosity must vanish.) The equation for the y-component of the momentum (3b) reduces to $\partial p/\partial y = 0$, and (3c) remains. If $2a > b$, we obtain the Euler equations, except that the equation for the y-momentum still reduces to $\partial p/\partial y = 0$. We note that in the third case and in the second case with $b > 1$, we have $u = o(1)$ and, by the argument given above, in the Euler limit we obtain zero velocity in the wake. This implies that the limiting Euler flow is of the Kirchhoff type and that $a = \frac{1}{2}$.

Finally, we observe that while the expansion for large x and fixed R (see [9]) may be useful for numerical calculations, the Reynolds number enters in such a way that successive terms are not successively smaller if x and R tend to infinity at the same rate. Hence a partial sum may not be used in an asymptotic expansion for large R for the region downstream of the wake bubble. The inner wake equations of [9] would have to be replaced by nonlinear boundary layer equations, probably with zero pressure gradient. In this region, we must assume u to be of order unity. However, in the Euler limit, the whole region, for $|y|$ not large, will disappear at infinity, so that we may not conclude that u remains of order unity in the Euler limit for moderate values of $|y|$. It may, for instance, tend to zero as in the free streamline solution.

To resolve the various questions about flow at large R, further analytic work is needed. Numerical investigations could provide guidance although present numerical methods do not allow us to study reasonably large Reynolds numbers. There are, however, various simplified problems the study of which could shed light on the problem of the magnitude of u in the wake.

[3] Various theoretical considerations indicate that if the limiting drag is nonzero and finite, the length must be of order R and its width $\ll R$.

This paper has touched on very few aspects of Goldstein's contributions to fluid dynamics. It would have been written the same way if the same subject matter had been given but if the occasion had not been a symposium in honor of Sydney Goldstein. However, quite independent of the occasion, one thing emerges clearly, namely the pioneering aspect of Goldstein's research. Instead of merely solving more special cases of Prandtl's equations, his papers on the wake behind a finite flat plate and on the singularity at the point of vanishing skin friction contained significant advances in the asymptotic theory of Navier–Stokes solutions and have stimulated further important research, some of which has been discussed above.

Acknowledgment. The author wishes to thank W. M. Kinnersley and A. F. Messieter for helpful discussions.

REFERENCES

[1] L. PRANDTL, *Über Flüssigkeiten bei sehr kleiner Reibung*, Verh. III Internat. Math. Kongr., Heidelberg, 1904; Teubner, Leipzig, 1905, pp. 484–491; English transl., Tech. Memo. 452, NACA, 1928.

[2] S. KAPLUN, *Low Reynolds number flow past a circular cylinder*, J. Math. Mech., 6 (1957), pp. 595–603; reprinted in [3].

[3] ———, *Fluid Mechanics and Singular Perturbations*, P. A. Lagerstrom, L. N. Howard and C. S. Liu, eds., Academic Press, New York, 1967.

[4] P. A. LAGERSTROM AND R. G. CASTEN, *Basic concepts underlying singular perturbation techniques*, SIAM Rev., 14 (1972), pp. 63–190.

[5] G. K. BATCHELOR, *On steady laminar flow with closed streamlines at large Reynolds number*, J. Fluid Mech., 1 (1952), pp. 177–190.

[6] R. P. FEYNMAN AND P. A. LAGERSTROM, *Remarks on high Reynolds number flows in finite domains*, Proc. IX International Congress on Applied Mechanics, vol. 3, Brussels, 1956.

[7] W. W. WOOD, *Boundary layers whose streamlines are closed*, J. Fluid Mech., 2 (1957), pp. 77–87.

[8] P. A. LAGERSTROM AND J. D. COLE, *Examples illustrating expansion procedures for the Navier–Stokes equations*, J. Rational Mech. Anal., 4 (1955), pp. 817–882.

[9] I. D. CHANG, *Navier–Stokes solutions at large distances from a finite body*, J. Math. Mech., 10 (1961), pp. 811–876.

[10] S. GOLDSTEIN, *Lectures on Fluid Mechanics*, Interscience, New York, 1960.

[11] ———, *On the two-dimensional steady flow of a viscous fluid behind a solid body*, Proc. Roy. Soc. London Ser. A., 142 (1953), pp. 545–573.

[12] M. VAN DYKE, *Higher approximations in boundary-layer theory. Part 3: Parabola in uniform stream*, J. Fluid Mech., 19 (1964), pp. 145–159.

[13] S. GOLDSTEIN, *Concerning some solutions of the boundary layer equations in hydrodynamics*, Proc. Cambridge Philos. Soc., 26 (1930), pp. 1–30.

[14] K. STEWARTSON, *On the flow near the trailing edge of a flat plate. II*, Mathematika, 16 (1969), pp. 106–121.

[15] A. F. MESSITER, *Boundary layer flow near the trailing edge of a flat plate*, SIAM J. Appl. Math., 18 (1970), pp. 241–257.

[16] S. GOLDSTEIN, *On laminar boundary-layer flow near a position of separation*, Quart. J. Mech. Appl. Math., 1 (1948), pp. 43–69.

[17] P. A. LAGERSTROM, *Laminar Flow Theory*, *High Speed Aerodynamics and Jet Propulsion*, vol. IV, F. K. Moore, ed., Princeton University Press, Princeton, N.J., 1964.

[18] K. STEWARTSON, *The behaviour of a laminar compressible boundary layer near a point of zero skin-friction*, J. Fluid Mech., 12 (1962), pp. 117–128.

[19] J. BUCKMASTER, *The behaviour of a laminar compressible boundary layer on a cold wall near a point of zero skin friction*, Ibid., 44 (1970), pp. 237–247.

[20] G. K. BATCHELOR, *A proposal concerning laminar wakes behind bluff bodies at large Reynolds number*, Ibid., 1 (1956), pp. 388–398.

[21] V. V. SYCHEV, *On laminar separation*, Mekhanika Zhidosti i Gaza, (1972), no. 3, pp. 47–59.

[22] A. F. MESSITER AND R. L. ENLOW, *A model for laminar boundary layer flow near a separation point*, this Journal, 25 (1973), pp. 655–670.